Problem 30. Let $E : \mathcal{B}(\mathbb{R}) \to B(H)$ be a spectral measure. Show that the following hold for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$:
(a) $B_1 \subseteq B_2 \implies E_{B_1} \leq E_{B_2}$,
(b) $E_{B_1} E_{B_2} = E_{B_2} E_{B_1}$,
(c) $E_{B_1 \cup B_2} = E_{B_1} \vee E_{B_2}$.

Problem 31. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function and

$$M_f : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$
$$M_f g(t) = f(t)g(t),$$

the (self-adjoint) multiplication operator. Determine the spectral projection $E(B)$ for $B \in \mathcal{B}(\mathbb{R})$.

Problem 32. Let $P \in B(H)$ be an orthogonal projection and

$$\{P\}' := \{T \in B(H) \mid TP = PT\},$$

the *commutant* of P. Show that

$$\{P\}' = \{T \in B(H) \mid \operatorname{ran} P \text{ and } \ker P \text{ are invariant under } T\}.$$

Problem 33. A vector $x \in H$ is called *cyclic* for an operator $T \in B(H)$ if $\mathcal{L}\{T^n x \mid n \in \mathbb{N}_0\}$ is dense in H. Let $A \in B(H)$ be a self-adjoint operator and x a cyclic vector. Denote the spectral measure at x by $\mu = \mu_x$ so that $\mu(B) = \langle E(B)x, x\rangle$. Then the map

$$U : L^2(\mu) \to H,$$
$$f \mapsto f(A)x,$$

is an isometry and

$$\left(UAU^{-1}f\right)(t) = M_{f(t)} \text{ on } L^2(\mu).$$